GCSE Maths – Geometry and Measures

Geometric Arguments and Proof (Higher)

Notes

WORKSHEET

Geometric Arguments and Proof (Higher)

A **proof** is a reasoned **argument** or **evidence** that makes certain the **truth** of a statement. Proofs must be constructed mathematically through a series of **logical and clearly explained steps**, that demonstrate to the reader that your evidence is, without any ambiguity or doubt, true.

There are three parts to a proof that you must include when answering a question:

1. **Working out** – show your logical steps and annotate them with operations if necessary.
2. **Statement** – write the true statement at the end of your working.
3. **Conclusion** – write a short sentence that describes your answer. You could also state the type of proof you have carried out. E.g., "Therefore, n is proven true by exhaustion."

Types of Proof

There are different ways to prove a statement, and you should choose the simplest possible method to apply to the question.

Proof by Exhaustion

To prove by exhaustion, you give an example of every possible scenario in which the statement is true. For example, consider the following question:

$$\textit{Prove that for every even number } n \textit{ where } n \leq 10, n^2 \textit{ is even.}$$

To prove this problem, you should list the even numbers 2, 4, 6, 8, 10 and list their squares 4, 16, 36, 64, 100. As the squares are all even, the statement is true. Write this in your conclusion and include the type of proof that you used.

You should only use **proof by exhaustion** when there are a small, **limited** number of **possibilities**.

Example: Prove that for every regular polygon with n sides where $3 \leq n \leq 6$ the value of each interior angle is less than $150°$.

Since $3 \leq n \leq 6$, n *can be* $n = 3$, $n = 4$, $n = 5$ *or* $n = 6$.

Since there are only 4 polygons described here, we can individually list their interior angles and show that they are less than $150°$*:*

$$\textit{Triangle interior angle}: \mathbf{60°} \quad 60 < 150$$
$$\textit{Square interior angle}: \mathbf{90°} \quad 90 < 150$$
$$\textit{Pentagon interior angle}: \mathbf{108°} \quad 108 < 150$$
$$\textit{Hexagon interior angle}: \mathbf{120°} \quad 120 < 150$$

Write a conclusion: We have proven by exhaustion that the statement is true.

Proof by Counterexample

To prove by **counterexample**, you give an example that directly **contradicts** the statement. You should prove by counterexample when there is an obvious contradiction. For example, consider the following question:

$$\textit{Prove that every multiple of } 7 \textit{ is odd.}$$

You should prove this statement by counterexample. For example, you know that $7 \times 10 = 70$, which is even. Write this in your conclusion and include the type of proof that you used.

Algebraic Proof

Proofs are most commonly completed algebraically. This involves showing, through a series of steps that use variables to represent values, that the statement is true beyond doubt.

Some algebraic substitutions are commonly included in proof.

Statement	Substitution
A number.	n
An even number.	$2n$
An odd number.	$2n + 1$
An even number squared	$(2n)^2$
An odd number squared.	$(2n + 1)^2$
Consecutive integers.	$n,\ n+1,\ n+2,\ n+3, \ldots$

For example, consider the following question:

$$\textit{Prove that the sum of three consecutive integers is a mulitple of } 3.$$

It would be impossible to demonstrate every possible example. Instead, you should use the substitutions for consecutive integers and complete the proof algebraically.

Example: Prove that the sum of three consecutive integers is always a multiple of 3.

1. Substitute the formula for consecutive integers.

   The sum of three consecutive integers can be represented by:

   $$(n) + (n+1) + (n+2)$$

2. Complete the proof by expanding the brackets and rearranging.

   $$(n) + (n+1) + (n+2) = n + n + n + 1 + 2 = 3n + 3 = 3(n+1)$$

   Since $3(n+1)$ *is divisible by* 3, *we have shown that any value of* n *will produce a multiple of 3.*

3. Form a conclusion.

   Therefore, the sum of any three consecutive integers is always a multiple of 3.

Geometric Proof

Proof in geometry usually involves vectors. You should be familiar with vector notation where bold type or arrows such as $\boldsymbol{a} = \overrightarrow{OA}$ represent the vector from the origin to point A.

Parallel Lines

Vector methods can be used to prove that two lines are **parallel**. Lines are parallel if their vectors are multiples of each other.

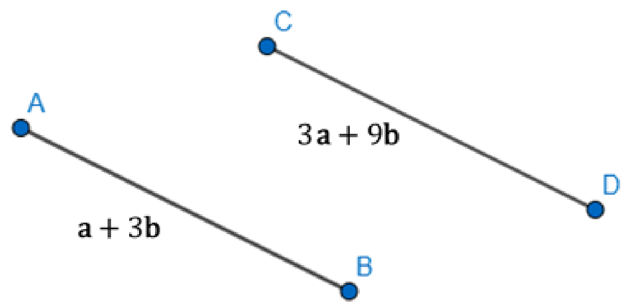
The lines AB and CD are parallel.

We can prove this using vector notation:

$$\overrightarrow{AB} = \mathbf{a} + 3\mathbf{b}$$

$$\overrightarrow{CD} = 3\mathbf{a} + 9\mathbf{b} = 3(\mathbf{a} + 3\mathbf{b})$$

$$\overrightarrow{AB} = 3(\overrightarrow{CD})$$



Therefore, the lines AB and CD are parallel because CD is a multiple of AB.

Collinear Points

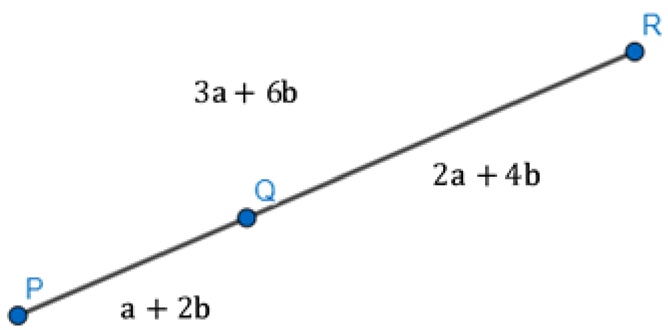
Points are **collinear** if they lie on the same **straight line**. To show that three points A, B and C are collinear, you need to show that the vectors between them are **all parallel**, such that AB is a multiple of AC and CB. In addition, the vectors between collinear points must share a **common point**.

$$\overrightarrow{PR} = 3\mathbf{a} + 6\mathbf{b}$$

$$\overrightarrow{PQ} = \mathbf{a} + 2\mathbf{b}$$

$$\overrightarrow{QR} = 2\mathbf{a} + 4\mathbf{b}$$

$$\overrightarrow{PR} = 3\left(\overrightarrow{PQ}\right) = \frac{3}{2}\left(\overrightarrow{QR}\right)$$

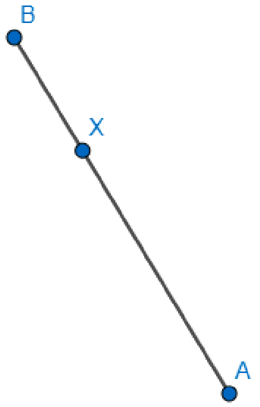


Therefore, the lines points P, Q and R are collinear because their vectors are multiples of one another, and they share a common point Q.

Vectors in Ratios

If point X splits a line AB in the ratio p : q, then $\overrightarrow{AX} = \frac{p}{p+q}(\overrightarrow{AB})$ and $\overrightarrow{XB} = \frac{q}{p+q}(\overrightarrow{AB})$.

Consider the following line AB where $\overrightarrow{AB} = 4\mathbf{a} + 12\mathbf{b}$. The point X divides the line AB in the ratio 3 :1.



There are 4 parts to the ratio, so $p + q = 4$.

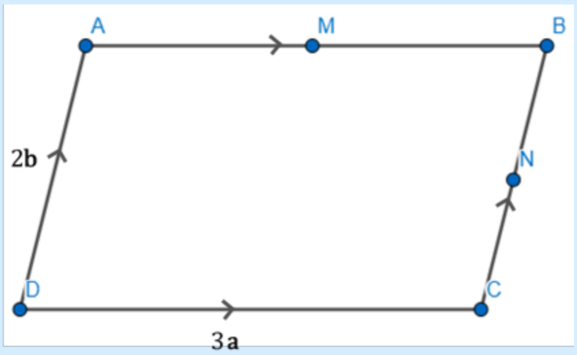
$$\overrightarrow{AX} = \frac{3}{4}(4\mathbf{a} + 12\mathbf{b}) = 3\boldsymbol{a} + 9\boldsymbol{b}$$

$$\overrightarrow{XB} = \frac{1}{4}(4\mathbf{a} + 12\mathbf{b}) = \boldsymbol{a} + 4\boldsymbol{b}$$

Example: ABCD is a parallelogram. M is the midpoint of AB and N is the midpoint of BC.

a) Write $\overrightarrow{AC}$ in terms of **a** and **b**.

b) Prove that $\overrightarrow{AC}$ is parallel to $\overrightarrow{MN}$



a) Find the vector $\overrightarrow{AC}$.

To get from point A to point C, we must go backwards along $\overrightarrow{DA}$ and forwards along $\overrightarrow{DC}$.

$$\overrightarrow{AC} = -(\overrightarrow{DA}) + (\overrightarrow{DC}) = -(2\boldsymbol{b}) + (3\boldsymbol{a}) = 3\boldsymbol{a} - 2\boldsymbol{b}$$

b) First, find the vector $\overrightarrow{MN}$.

To get from point M to point N, we must go forwards along half of $\overrightarrow{AB}$ and backwards along half of $\overrightarrow{CB}$:

$$\overrightarrow{MN} = \frac{1}{2}(\overrightarrow{AB}) - \frac{1}{2}(\overrightarrow{CB}) = \frac{1}{2}(3\boldsymbol{a}) - \frac{1}{2}(2\boldsymbol{b}) = \frac{3}{2}\boldsymbol{a} - \boldsymbol{b}$$

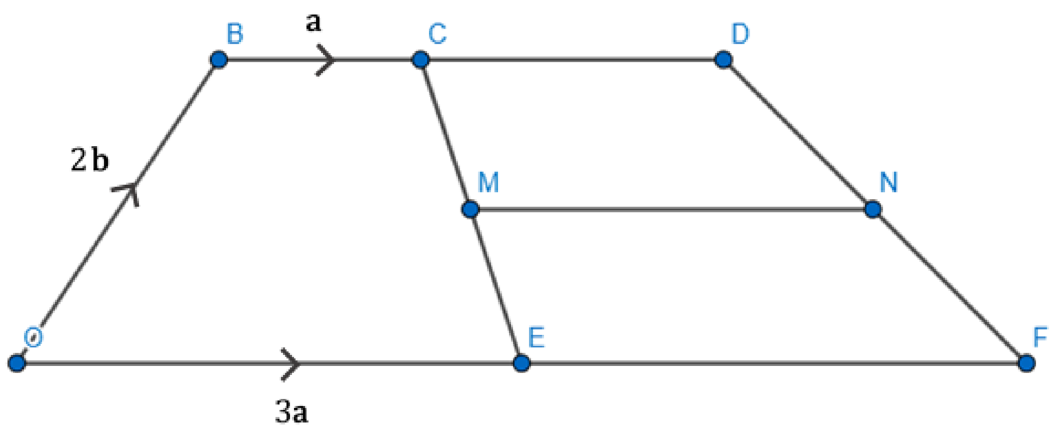
Show that $\overrightarrow{AC}$ and $\overrightarrow{MN}$ are parallel by writing one as a multiple of the other:

$$\frac{3}{2}\boldsymbol{a} - \boldsymbol{b} = \frac{1}{2}(3\boldsymbol{a} - 2\boldsymbol{b})$$

$\overrightarrow{MN} = \frac{1}{2}(\overrightarrow{AC})$ $\therefore$ *$\overrightarrow{AC}$ and $\overrightarrow{MN}$ are parallel.*

Geometric Arguments and Proof – Practice Questions

1. OBDF is a trapezium.
   $\overrightarrow{BC} = \mathbf{a}$, $\overrightarrow{OB} = 2\mathbf{b}$ and $\overrightarrow{OE} = 3\mathbf{a}$.
   C is the midpoint of $\overrightarrow{BD}$ and E is the midpoint of $\overrightarrow{OF}$.
   M is the midpoint of $\overrightarrow{CE}$ and N is the midpoint of $\overrightarrow{DF}$.



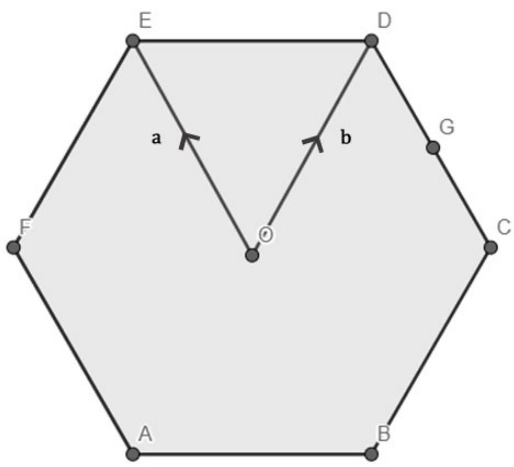
a) Find the vector $\overrightarrow{CE}$ in terms of **a** and **b**.
b) Find the vector $\overrightarrow{DF}$ in terms of **a** and **b**.
c) Prove that $\overrightarrow{MN}$ is parallel to $\overrightarrow{OF}$.

2. ABCDEF is a regular hexagon with centre O.

$$\overrightarrow{OE} = \boldsymbol{a}$$

$$\overrightarrow{OD} = \boldsymbol{b}$$

   The point X lies on an extension of ED, such that $\overrightarrow{EX} : \overrightarrow{DX} = 2 : 1$ and $\overrightarrow{EX} = -2a + 2b$.
   G is the midpoint of CD.



a) Draw the vector $\overrightarrow{AO}$ and label it in terms of **a** and **b**.

b) Label all 6 sides of the hexagon in terms of **a** and **b**.

c) Find the vector $\overrightarrow{DG}$ in terms of **a** and **b**.

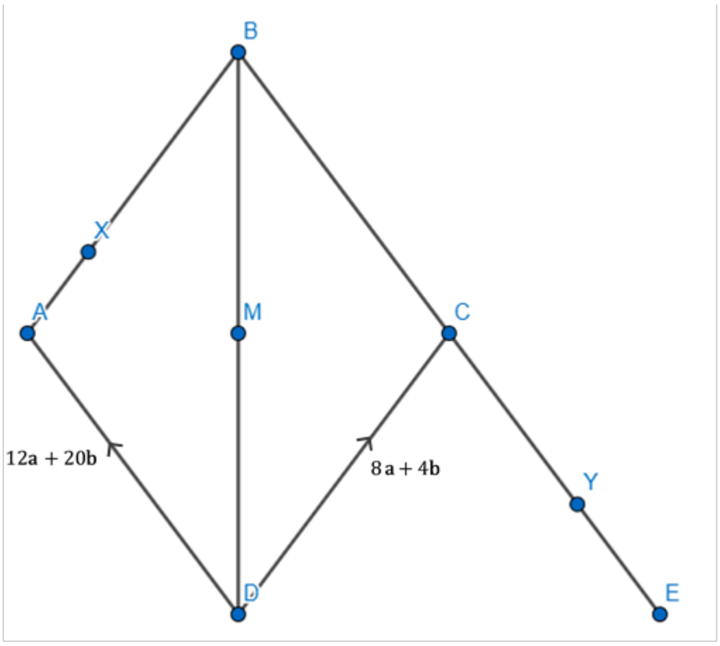
d) Find the vector $\overrightarrow{DX}$ in terms of **a** and **b**.

e) Hence, prove that O, G and X lie on a straight line.

3. ABCD is a parallelogram.

$\overrightarrow{DA} = 12\mathbf{a} + 20\mathbf{b}$ and $\overrightarrow{DC} = 8\mathbf{a} + 4\mathbf{b}$.
X lies on the line $\overrightarrow{AB}$ such that $\overrightarrow{AX} : \overrightarrow{XB} = 1 : 3$.
M is the midpoint of $\overrightarrow{DB}$.
$\overrightarrow{CE}$ is an extension of $\overrightarrow{BC}$.
Y lies on the line $\overrightarrow{CE}$ such that $\overrightarrow{CY} = -\frac{1}{2}\overrightarrow{DA}$.

Prove that X, M and Y are collinear.



Worked solutions for the practice questions can be found amongst the worked solutions for the corresponding worksheet file.